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HEAT CONDUCTION IN TECHNOLOGICAL PROCESSES

PROPAGATION OF HEAT IN THE SPACE AROUND A CYLINDRICAL SURFACE AS A NON-MARKOVIAN RANDOM PROCESS

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Consideration has been given to the propagation of heat in the space around a cylindrically shaped body in the presence of fluctuations of the heat flux through its surface. It has been shown that the corresponding random changes in the temperature and the heat flux are described by integral stochastic equations and belong to the class of non-Markovian processes. Statistical characteristics of the considered fluctuations, including one-dimensional and multidimensional characteristic functions, spectral densities, and probability densities, have been found.

Keywords: *heat conduction, non-Markovian process, integral stochastic equations.*

Introduction. Heat-conduction processes occurring in physical media are accompanied by fluctuations of their parameters (temperature and heat flux) due to the random changes in the power of the heat source, the thermal-conductivity coefficient of the medium and the intensity of the thermodynamic flow in it, and for other reasons. Physical processes giving rise to such fluctuations can be exemplified by current noise in conductors and semiconductors [1] and by the flicker noise of the material's electrical conduction [2]. One generally describes the propagation of heat in a medium using differential heat-conduction equations with corresponding initial and boundary conditions and takes account of fluctuations of its temperature and the heat fluxes in it by adding random functions to the obtained differential equations. Such an approach enables one to use a well-developed theory of stochastic differential systems [3]. Random changes in the physical quantities used in the formulation of such a problem represent Markovian processes.

Actual physical media, however, possess hereditary properties which can be characterized with differential operators only approximately. Although such an approximation may be reckoned as satisfactory, it often becomes necessary to more generally investigate the heat-conduction phenomenon for taking account of the non-Markovian character of fluctuations of temperatures and heat fluxes. It is proposed that the corresponding integral equations [4, 5] whose kernels can in principle allow for the hereditary properties of a non-Markovian heat-conduction process be used instead of differential stochastic equations.

Integral transformations have been used in [6] for finding analytical solutions of boundary-value problems of heat conduction in an infinite domain bounded by a cylindrical surface. The method of solution of nonlinear nonstationary heat-conduction problems using nonlinear Volterra integral equations of the second kind has been proposed in [7].

We note that the indicated approach to description of Brownian motion, the evaporation of a liquid droplet in the atmosphere, and the motion of a particle in a medium with a fluctuating friction factor makes it possible to substantially refine results obtained with classical methods [8, 9].

In this work, consideration is given to the process of propagation of heat in the medium around an infinite cylindrical surface the heat flux through which can randomly be changed with time. It is shown that this phenomenon is described by the linear Volterra integral equation of the second kind, and the corresponding functions are non-Markovian. Statistical characteristics describing the problem of physical quantities, among which are characteristic functions, spectral densities, and probability densities, are found using the method of description of non-Markovian random processes prescribed by the linear integral transformations.

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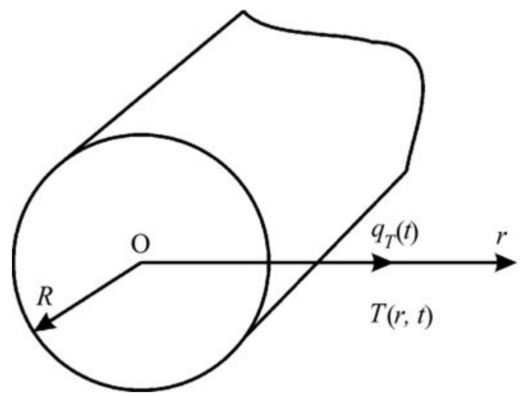


Fig. 1. Heat conduction in the space around the cylindrical surface.

Formulation of the Problem. We consider an infinitely long cylindrical body of radius R manufactured from a material with a high conductivity, heat capacity per unit volume C_V, and thermal diffusivity χ_c; the body is in an unbounded heat-conducting medium with a thermal conductivity κ and a thermal diffusivity χ (Fig. 1). We will assume that the surface temperature of the cylinder, at a given instant of time, is everywhere the same and is a certain function of time: $T(R, t) = T_R(t)$. We note that in actual practice this condition corresponds to the case of consideration of the heat-conduction phenomenon in the medium around a cylindrical body of small radius at a distance smaller or of the order of R from its surface. Also, we will assume that at the initial instant of time, the temperature throughout the space, including the temperature of the cylinder material, is the same and equal to zero.

Within the framework of the assumptions made, the temperature of the medium outside the cylinder is dependent just on the distance r to its axis of symmetry and obeys a differential heat-conduction equation of the form

$$\frac{\partial T(r,t)}{\partial t} = \chi \left(\frac{\partial^2 T(r,t)}{\partial r^2} + \frac{1}{r} \frac{\partial T(r,t)}{\partial r} \right), \quad r > R, \tag{1}$$

with the initial and boundary conditions

$$T(r,t)\big|_{t=0} = 0, \tag{2}$$

$$T(r,t)\Big|_{\substack{r=R,\\ t>0}} = T_R(t). \tag{3}$$

Stochastic Integral Equation. It can be shown that if the heat $Q(\tau)$ is instantaneously released, at a certain instant of time τ, on each unit length of the cylindrical surface in question, the medium's temperature for $r > R$ at the instant of time t will be determined by the expression [10]

$$T(r,t) = \frac{\chi}{\kappa} Q(\tau)\, G(r, R, t-\tau), \tag{4}$$

where $G(r, R, t-\tau)$ is the influence function of the instantaneous cylindrical heat source (Green's function) defined as

$$G(r, R, t-\tau) = \frac{1}{4\pi\chi(t-\tau)} \exp\left(-\frac{r^2 + R^2}{4\chi(t-\tau)} \right) I_0 \left(\frac{Rr}{2\chi(t-\tau)} \right). \tag{5}$$

If $Q(\tau)\delta(t-\tau) = 2\pi R q_T(t)$, in the case of an arbitrary heat flux through the surface of the cylinder $q_T(t)$ the temperature at a distance r from its axis of symmetry will be determined from the expression

$$T(r,t) = \frac{R}{2\kappa} \int_0^t \frac{q_T(\tau)}{t-\tau} \exp\left(-\frac{r^2+R^2}{4\chi(t-\tau)}\right) I_0\left(\frac{Rr}{2\chi(t-\tau)}\right) d\tau \,. \tag{6}$$

We find the derivative of $\frac{\partial T(r,t)}{\partial r}$, using expression (6) and the fact that the derivative of the Bessel function of zero order is equal in argument to the Bessel function of first order: $\frac{dI_0(x)}{dx} = I_1(x)$. We obtain

$$\frac{\partial T(r,t)}{\partial r} = \frac{R}{4\chi\kappa} \int_0^t \frac{q_T(\tau)}{(t-\tau)^2} \exp\left(-\frac{r^2+R^2}{4\chi(t-\tau)}\right) \left[R I_1\left(\frac{Rr}{2\chi(t-\tau)}\right) - r I_0\left(\frac{Rr}{2\chi(t-\tau)}\right) \right] d\tau \,. \tag{7}$$

The heat flux through the surface of the cylindrical body $q_T(t)$ is determined by the general relation

$$q_T(t) = -\kappa \left. \frac{\partial T(r,t)}{\partial r} \right|_{r=R} + \xi_{q_T}(t) \,, \tag{8}$$

where the function $\xi_{q_T}(t)$ represents a random heat flux whose statistical properties are determined by the character of the fluctuation source; the mean value is $\langle \xi_{q_T}(t) \rangle = 0$. At the same time, such a flux can also be determined according to the equation

$$q_T(t) = -\frac{C_V R}{2} \frac{dT_R(t)}{dt} \,. \tag{9}$$

It should be emphasized that expression (9) holds true for the case of the assumed presence of the high thermal conductivity of the cylinder's material compared to the conductivity of the medium, i.e., on condition that $\chi << \chi_c$. From the obtained relation (7), with account of formulas (8) and (9), we find

$$\frac{C_V R}{2} \int_0^t K(t-\tau) Z(\tau) \, d\tau = \xi_{q_T}(t) \,, \tag{10}$$

where

$$Z(t) = \frac{dT_R(t)}{dt} \tag{11}$$

and

$$K(t-\tau) = \delta(t-\tau) + \frac{R^2}{4\chi(t-\tau)^2} \exp\left(-\frac{R^2}{2\chi(t-\tau)}\right) \left[I_1\left(\frac{R^2}{2\chi(t-\tau)}\right) - I_0\left(\frac{R^2}{2\chi(t-\tau)}\right) \right]. \tag{12}$$

Thus, the process of propagation of heat in the space around the cylindrical surface of radius R in the presence of the random heat flux through it is described by the stochastic Volterra integral equation of the second kind (10). The random process described by the integral equations of the form (10) with nonexponential kernels is not reduced to a system of differential stochastic equations and belongs to the class of non-Markovian processes [5]. Consequently, fluctuations of the quantity $Z(t)$ and changes in the temperature of the cylinder surface $T_R(t)$ and the flux $q_T(t)$ possess hereditary properties: their statistical characteristics beginning with a certain time τ_0 are dependent on the behavior of these functions at $t < \tau_0$.

We note that for large values of the radius R, expression (12) can be written using the approximate formula [11]

$$K(t-\tau) = \delta(t-\tau) + \frac{\sqrt{\chi}}{R\sqrt{\pi(t-\tau)}} \,, \tag{13}$$

whose substitution into Eq. (10) instead of relation (12) yields an integral equation of the form

$$\frac{C_V R}{2} Z(t) + \frac{\kappa}{\sqrt{\pi\chi}} \int_0^t \frac{1}{\sqrt{t-\tau}} Z(\tau)\, d\tau = \xi_{q_T}(t) , \quad (14)$$

this equation has been investigated in [12] as applied to the phenomenon of heat conduction in the half-space above a flat surface. Indeed, let us consider a cylindrical shell of radius R and such a thickness h that $h << R$. Then, instead of Eq. (5), we obtain the relation $q_T(t) = -C_V h \frac{dT_R(t)}{dt} + \xi_{q_T}(t)$. Taking into account the fact that the product $C_V h$ represents the heat capacity of unit surface of such a shell C_S, we should take the value of C_S instead of $\frac{C_V R}{2}$ in (14); this precisely leads to the integral equation considered in [12] and describing heat conduction in a half-space bounded by a flat surface.

It is noteworthy that if the main source of heat conduction is heat transfer, the upper limit of integration in Eq. (10) should, generally speaking, be taken to be $t - \delta t$, where δt is the small parameter equal to the time of free motion of the particles of the heat-conducting medium in order of magnitude.

Let us elucidate the statistical properties of the introduced random heat flux $\xi_{q_T}(t)$ through the surface of the cylindrical body. The function $\xi_{q_T}(t)$ in many cases can be represented as white noise with an intensity ν, which is bounded above by a certain limiting frequency ω_{max}; this frequency can be evaluated accurate to a constant using the formula

$$\omega_{max} \sim \frac{\chi_c}{R^2} . \quad (15)$$

To evaluate ω_{max} we will assume that the cylinder has been manufactured from copper (for which the thermal diffusivity is $\chi_c = 10^{-4}$ m^2/s) and having radius $R = 10$ μm. We obtain that $\omega_{max} \sim 10^6$ s^{-1}. Thus, white noise characteristic of fluctuations of the heat flux through the cylinder surface turns out to be wide-band, as far as the spectrum is concerned. Evaluation of the intensity ν can be carried out from the formula

$$\nu \sim \frac{\kappa}{R^3} k_B T^2 . \quad (16)$$

The obtained estimate follows from relations (8) and (9) which, upon the replacement of the derivative $\partial T/\partial r$ by the ratio T_R/R, lead to an equation of the form $\frac{C_V R}{2}\frac{dT_R(t)}{dt} + \frac{\kappa}{R} T_R(t) = \frac{2}{C_V R}\xi_{q_T}(t)$ or $T_R(t) + \frac{2\kappa}{C_V R^2} T_R(t) = \frac{4}{C_V^2 R^2}\xi_{q_T}(t)$, which corresponds to the Langevin equation for Brownian motion. The last formula makes it possible to find [13] an estimate of the intensity of the Langevin source $\xi_{q_T}(t)$ and is represented by relation (16). At the temperature $T_0 = 300$ K, the estimate given by expression (16) leads to an intensity of $\nu \sim 10^4$ J^2/(m^4·s) for the copper cylindrical body of radius $R = 50$ nm placed in water.

Case of the White Noise of Fluctuations of the Heat Flux through the Cylindrical Surface of Small Radius. We consider heat conduction in the medium around the cylindrical shell in the case where the parameter $\frac{R^2}{\chi} << 1$, which (as is seen from expression (15)) corresponds to the case of a very high boundary frequency of the spectrum of fluctuations of the heat flux through the cylindrical surface (white noise). Also, setting $t - \tau > \delta t$, we obtain, upon the series expansion of modified Bessel functions and exponents and retention of the first terms of expansion, that

$$Z(t) - \frac{R^2}{4\chi} \int_0^t \frac{1}{(t-\tau)^2} Z(\tau)\, d\tau = \frac{2}{C_V R} \xi_{q_T}(t) . \quad (17)$$

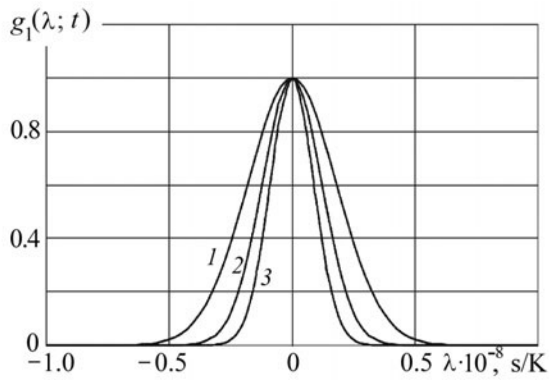


Fig. 2. Function $g_1(\lambda;\ t)$ vs. parameter λ at different instants of time: 1) $t = 10^{-7}$ s, 2) 10^{-6} s, and 3) ∞.

The solution of the Volterra integral equation of the second kind (17) in the general case can be written as [14]

$$Z(t) = \frac{2}{C_V R} \int_0^t [\delta(t-\tau) + F(t-\tau)]\, \xi_{q_T}(\tau)\, d\tau \,, \tag{18}$$

where the resolvent $F(t-\tau)$ is determined by the recurrence relation

$$F(t-\tau) = \sum_{n=1}^{\infty} F_n(t-\tau) \,. \tag{19}$$

Here we have

$$F_1(t-\tau) = \frac{R^2}{4\chi} \frac{1}{(t-\tau)^2} \,, \tag{20}$$

$$F_n(t-\tau) = \int_{\tau+\delta t}^{t-\delta t} F_1(t-s)\, F_{n-1}(s-\tau)\, ds \,, \quad n > 1 \,. \tag{21}$$

The above condition $\frac{R^2}{\chi} << 1$ makes series (19) rapidly convergent. On condition that $\frac{R^2}{\chi} << \delta t$ (cylindrical surfaces of small radius, or thin filaments), we can disregard the second term and terms that follow in series (19) compared to the first term and finally obtain

$$F(t-\tau) = \frac{R^2}{4\chi} \frac{1}{(t-\tau)^2} \,. \tag{22}$$

Statistical Characteristics. The initial integral equation (18) for the case of the resolvent $F(t-\tau)$ of the form (22) enables us to find any statistical characteristics of the process $Z(t)$, if we use the method developed in [4] for description of non-Markovian random processes prescribed by the linear integral transformations. Thus, for the one-dimensional $g_1(\lambda;\ t)$ and multidimensional $g_L(\lambda_1,\ \ldots,\ \lambda_L;\ t_1,\ \ldots,\ t_L)$ characteristic functions of the process $Z(t)$ on condition that the random heat flux $\xi_{q_T}(t)$ represents white noise of intensity ν, we obtain

$$g_1(\lambda;\ t) = \exp\left[-\frac{1}{24} \frac{R^2 \nu \lambda^2}{\chi^2 C_V^2} \left(\frac{1}{\delta t^3} - \frac{1}{t^3}\right)\right], \tag{23}$$

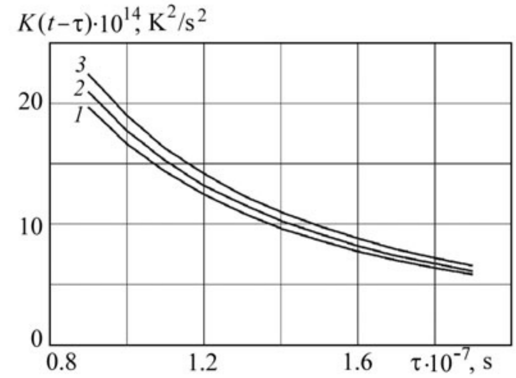


Fig. 3. Function $K(t-\tau,\ t)$ vs. parameter τ at different instants of time: 1) $t = 10^{-7}$ s, 2) 10^{-6} s, and 3) 1 s.

$$g_L(\lambda_1, ..., \lambda_L; t_1, ..., t_L) = \exp\left[-\frac{1}{24}\frac{R_2\nu}{\chi^2 C_V^2}\sum_{\substack{k,l=1,\\k<l}}^{L}\frac{\lambda_l\lambda_k}{(t_l-t_k)^2}\left[\frac{1}{\delta t}+\frac{t_k^2-t_l^2+t_lt_k}{t_lt_k(t_l-t_k)}+\frac{2}{t_l-t_k}\ln\frac{t_l\delta t}{t_k(t_l-t_k)}\right]\right]. \quad (24)$$

The function $g_1(\lambda;\ t)$ defined by Eq. (23) is presented graphically in Fig. 2. Here and in what follows we use as examples copper cylindrical bodies of small radius placed in water ($R = 10^{-8}$ m, $\nu = 10^4$ J^2/(m^4·s), $\chi = 2\cdot10^{-7}$ m^2/s, and $C_V = 1.3\cdot10^3$ J/(K·m^3)).

Expressions for the characteristic functions (23) and (24) enable us to find any statistical characteristics of the process $Z(t)$. For the mathematical expectation $\langle Z(t)\rangle$ and the variance $D_Z(t)$ of the process $Z(t)$, we find, from (23),

$$\langle Z(t)\rangle = \frac{\partial g_1(\lambda; t)}{i\partial\lambda}\bigg|_{\lambda=0} = 0\,, \quad (25)$$

$$D_Z(t) = \langle Z^2(t)\rangle = -\frac{\partial g_1(\lambda; t)}{\partial\lambda^2}\bigg|_{\lambda=0} = \frac{R^2\nu}{24\chi^2C_V^2}\left(\frac{1}{\delta t^3}-\frac{1}{t^3}\right). \quad (26)$$

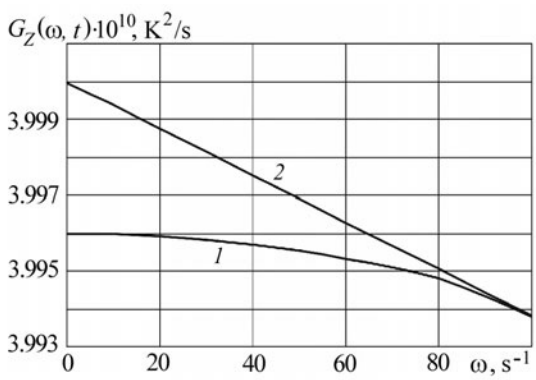
It is seen from the found formula (26) that the steady-state heat-conduction process ($t \to \infty$) is accompanied by a constant value of the variance $D_Z(t)$ of the rate of change in the temperature of the cylindrical surface $Z(t)$, which is dependent on the parameters of the body and the medium and on the characteristic quantity δt considered earlier:

$$D_Z(t)\big|_{t\to\infty} = \frac{R^2\nu}{24\chi^2C_V^2\delta t^3}\,. \quad (27)$$

For the correlation function $K(t_1,\ t_2) = \langle Z(t_1)Z(t_2)\rangle$, with the two-dimensional characteristic function $g_2(\lambda_1, \lambda_2;\ t_1,\ t_2)$ determined from (24), we obtain

$$K(t_1, t_2) = -\frac{\partial g_2(\lambda_1, \lambda_2; t_1, t_2)}{\partial\lambda_1\partial\lambda_2}\bigg|_{\substack{\lambda_1=0,\\ \lambda_2=0}} = \frac{1}{48}\frac{R_2\nu}{\chi^2C_V^2}\frac{1}{(t_2-t_1)^2}\left[\frac{1}{\delta t}+\frac{t_1^2-t_2^2+t_1t_2}{t_1t_2(t_2-t_1)}+\frac{2}{t_2-t_1}\ln\frac{t_2\delta t}{t_1(t_2-t_1)}\right]. \quad (28)$$

Dependence (28) yields that at $t_1 \to \infty$ and $t_2 \to \infty$ (steady-state heat-conduction process), the correlation function $K(t_1,\ t_2) = K(t_2 - t_1) = K(\tau)$ takes the form



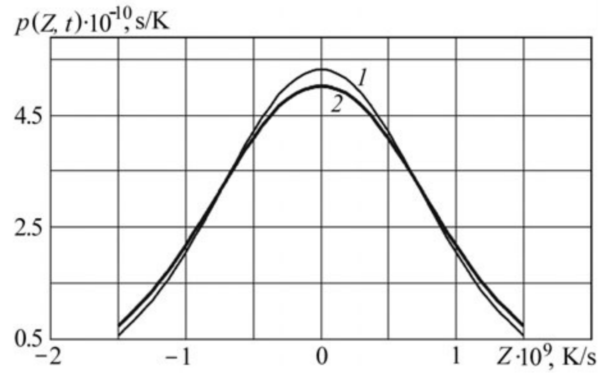


Fig. 4. One-sided spectral density $G_Z(\omega, t)$ vs. frequency of white noise at different instants of time: 1) 0.01 and 2) 1 s.

Fig. 5. Probability density $p(Z, t)$ vs. rate of change in the temperature of the cylinder surface at different instants of time: 1) $t = 10^{-7}$ and 2) 10^{-6} s.

$$K(\tau) = \frac{1}{48} \frac{R^2 \nu}{\chi^2 C_V^2} \frac{1}{\tau^2} \left[\frac{1}{\delta t} + \frac{1}{\tau} + \frac{2}{\tau} \ln \frac{\delta t}{\tau} \right]. \tag{29}$$

The function $K(t - \tau, t)$ defined by relation (28) is presented graphically in Fig. 3.

The found correlation function (28) enables us to find one-sided spectral density of the process $Z(t)$:

$$G_Z(\omega, t) = 2 \int_0^t K(t - \tau, t) \cos \omega\tau d\tau . \tag{30}$$

Figure 4 displays a result of numerical computation of the one-sided spectral density $G_Z(\omega, t)$ using formula (30) for different times t. In the particular case of small ω values and short t, with account of the condition $\delta t \ll t$, (28) and (30) yield that

$$G_Z(\omega, t) symbol_{\omega \to 0} = 2 \int_0^t K(t - \tau, t) \left(1 - \frac{\omega^2 \tau^2}{2}\right) d\tau = \frac{1}{48} \frac{R^2 \nu}{\chi^2 C_V^2} \frac{t}{\delta t} \left(\frac{4}{t \delta t} - \omega^2\right). \tag{31}$$

Let us find the expression for the one-dimensional probability density function $p(Z, t)$:

$$p(Z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_1(\lambda; t) \exp(-i\lambda Z)\, d\lambda = \sqrt{\frac{1}{2\pi D(t)}} \exp\left(-\frac{Z^2}{2D(t)}\right). \tag{32}$$

Figure 5 gives dependence (32) for different t. It is clearly seen that the plot of the probability density of $Z(t)$ fluctuations is "smeared" along the Z axis, tending to a stationary Gaussian curve at $t \to \infty$.

Conclusions. Study of the process of propagation of heat even in the case of a relatively simple model (infinite cylinder of constant radius fluctuations of the heat flux through whose surface are independent of a point on its surface) requires integral stochastic equations for statistical description, whereas the fluctuations themselves of the quantities should be considered as non-Markovian random processes. The obtained results are of importance for description of random fluctuations of the temperature of cylindrical bodies of micrometer and submicrometer radii in media with a low thermal conductivity.

NOTATION

C_S, heat capacity of unit surface of the cylinder; C_V, heat capacity per unit volume of the cylinder material; $I_0(x)$, modified Bessel function of zero order; k_B, Boltzmann constant; $q_T(t)$, heat flux through the surface of the cylindrical body; R, cylinder radius; r, distance to the axis of symmetry of the cylinder; t, time variable; $T_R(t)$, temperature of the cylinder surface; x, auxiliary variable; Z, rate of change in the temperature of the cylinder surface; κ, thermal conductivity of the medium around the cylinder; λ, parameter of the characteristic function; ν, intensity of fluctuations of the heat flux through the surface of the cylindrical body which have the form of white noise; τ, integration variable (time); χ, thermal diffusivity of the medium around the cylinder; χ_c, thermal diffusivity of the cylinder material; ω, white-noise frequency. Subscript: c, cylinder.

REFERENCES

1. M. Buckingham, *Noise in Electronic Devices and Systems* [Russian translation], Mir, Moscow (1986).
2. G. N. Bochkov and Yu. E. Kuzovlev, Innovations in investigations of the 1/*f* noise, *Usp. Fiz. Nauk*, **141**, 151–176 (1983).
3. V. S. Pugachev and I. N. Sinitsyn, *Stochastic Differential Systems* [in Russian], Nauka, Moscow (1990).
4. A. N. Morozov, A method for describing non-Markovian processes assigned by a linear integral transformation, *Vestn. MGTU*, *Estestv. Nauki*, No. 3, 47–56 (2004).
5. A. N. Morozov, *Irreversible Processes and Brownian Motion* [in Russian], Izd. MGTU im. N. É. Baumana, Moscow (1997).
6. É. M. Kartashov, A class of integral transformations for the generalized equation of nonstationary heat conduction, *Inzh.-Fiz. Zh.*, **81**, No. 1, 123–130 (2008).
7. V. D. Belik, B. A. Uryukov, G. A. Frolov, and G. V. Tkachenko, Numerical analytical method of solving the nonlinear nonstationary heat conduction equation, *Inzh.-Fiz. Zh.*, **81**, No. 6, 1058–1063 (2008).
8. A. N. Morozov and A. V. Skripkin, Application of integral transformations to description of the Brownian motion as a non-Markovian random process, *Izv. Vyssh. Uchebn. Zaved.*, *Fiz.*, No. 2, 66–74 (2009).
9. A. N. Morozov and A. V. Skripkin, Statistical description of an oscillator exposed to the action of a fluctuating friction factor, *Vestn. MGTU*, *Estestv. Nauki*, No. 2, 3–15 (2008).
10. A. N. Tikhonov and A. A. Samarskii, *Mathematical Physics Equations* [in Russian], Nauka, Moscow (1977).
11. A. F. Nikiforov and V. B. Uvarov, *Special Functions of Mathematical Physics* [in Russian], Nauka, Moscow (1984).
12. A. N. Morozov and A. V. Skripkin, Application of the Volterra second-kind equation to description of viscous friction and thermal conductivity, *Vestn. MGTU*, *Estestv. Nauki*, No. 3, 62–71 (2006).
13. Yu. L. Klimontovich, *Statistical Physics* [in Russian], Nauka, Moscow (1982).
14. V. Volterra, *Theory of Functionals and of Integral and Integro-Differential Equations* [Russian translation], Nauka, Moscow (1982).